Notes on anelastic effects and thermal noise in suspensions of test masses in interferometric gravitational-wave detectors

Constantin Brif
LIGO Project, California Institute of Technology, Pasadena, CA 91125

Contents

1 Introduction

The thermal noise is expected to be one of the main limiting factors on the sensitivity of interferometric gravitational-wave detectors like LIGO and VIRGO. Thermal fluctuations of internal modes of the interferometer's test masses and of suspension modes will dominate the noise spectrum at the important frequency range between 50 and 200 Hz (seismic noise and photon shot noise dominate for lower and higher frequencies, respectively). It is important to note that off-resonant thermal noise level in high-quality systems is so low that it is unobservable in table-top experiments. Therefore, predictions of the thermal-noise spectrum in LIGO are based on a combination of theoretical models (with the fluctuation-dissipation theorem of statistical mechanics serving as a basis) and experimental measurements of quality factors of systems and materials involved. It is assumed that losses in the test masses and suspensions will occur mainly due to internal friction in their materials, which is related to anelasticity effects in solids.

These informal notes comprise some basic results on the theory of anelasticity and thermal noise in pendulum suspensions. This collection is by no means complete and focus on aspects which are of interest for the author. The original results can be found in a number of books, research papers, and theses. Some of these sources are listed in a short bibliography at the end of the present text; a list of research papers (since 1990) devoted to various aspects of the thermal noise in interferometric gravitational-wave detectors was prepared by the author and is available as a separate document.

2 Fluctuation-dissipation theorem

Consider a linear one-dimensional mechanical system with coordinate $x(t)$. If a force $F(t)$ acts on the system, than in the frequency domain the force and the coordinate are related by

$$x(\omega) = H(\omega)F(\omega), \tag{2.1}$$

where $H(\omega)$ is the system response function. Then the spectral densities (see Appendix) are related by

$$S_x(\omega) = |H(\omega)|^2 S_F(\omega). \tag{2.2}$$

The impedance of the system is defined as $Z(\omega) = F(\omega)/v(\omega) = F(\omega)/[i\omega x(\omega)]$. Therefore, the impedance and the response function are related by $Z(\omega) = 1/[i\omega H(\omega)]$.

If the system is in equilibrium with the thermal bath of temperature $\mathcal{T}$, then the *fluctuation-dissipation theorem* (FDT) says that the spectral density of the thermal force is

$$S_F^{\rm th}(\omega) = 4k_B\mathcal{T}\mathrm{Re}Z(\omega), \tag{2.3}$$

where k_B is the Boltzmann constant. The form (2.3) of the FDT is valid in the classical regime, when the thermal energy $k_B\mathcal{T}$ is much larger than the energy quantum $\hbar\omega$. Using the FDT, one readily obtains the thermal noise spectrum

$$S_x^{\rm th}(\omega) = \frac{4k_B\mathcal{T}}{\omega^2}\mathrm{Re}Y(\omega), \tag{2.4}$$

where $Y(\omega) = 1/Z(\omega)$ is the admittance and $\mathrm{Re}Y(\omega) = \mathrm{Re}Z(\omega)/|Z(\omega)|^2$ is the conductance. The FDT is the basis for calculations of the thermal noise spectrum in interferometric gravitational-wave detectors.

2.1 Example: Damped harmonic oscillator

Consider a damping harmonic oscillator of mass m, spring constant k, and damping constant γ. The equation of motion is

$$m\ddot{x} + \gamma\dot{x} + kx = F(t). \tag{2.5}$$

In the frequency domain this can be written as

$$(-m\omega^2 + i\gamma\omega + k)x(\omega) = F(\omega). \tag{2.6}$$

The impedance of this system is $Z(\omega) = \gamma + i(m\omega - k/\omega)$. Then the FDT gives the spectrum densities:

$$S_F^{\rm th}(\omega) = 4k_B\mathcal{T}\gamma, \qquad S_x^{\rm th}(\omega) = \frac{4k_B\mathcal{T}\gamma}{(m\omega^2 - k)^2 + \gamma^2\omega^2}. \tag{2.7}$$

3 Anelasticity of solids

The FDT means that if a system has no dissipation channel, thermal fluctuations will be zero. For an ideal elastic spring without friction, $\mathrm{Re}Z(\omega) = 0$, and there are no fluctuations: $S_x^{\rm th}(\omega) = 0$. Deviations of solids from the ideal elastic behavior (anelasticity) will result in internal friction (dissipation) and related thermal noise. For gravitational-wave detectors like LIGO, the test masses will be highly isolated, so the internal friction in materials of which the masses and their suspensions are made is believed to be the main source of dissipation and thermal noise.

3.1 The complex Young modulus and the loss function

Deformations of solids are usually described in terms of stress σ and strain ϵ (equivalents of mechanical restoring spring force F_s and displacement x, respectively). Perfect elastic solids satisfy Hooke's law

$$\sigma(t) = E\epsilon(t), \tag{3.1}$$

where E is the (constant) Young modulus (an equivalent of the spring constant k). Anelasticity can be described by introducing the complex Young modulus (or the complex spring constant in a mechanical model). This is done in the frequency domain:

$$E(\omega) = \frac{\sigma(\omega)}{\epsilon(\omega)}, \qquad k(\omega) = \frac{F_s(\omega)}{x(\omega)}. \tag{3.2}$$

If an external force $F(t)$ acts on a point mass m attached to such an anelastic spring, then the equation of motion in the frequency domain is

$$[-m\omega^2 + k(\omega)]x(\omega) = F(\omega). \tag{3.3}$$

The impedance of this system is

$$Z(\omega) = \frac{-m\omega^2 + k(\omega)}{i\omega}, \tag{3.4}$$

and $\mathrm{Re}\, Z(\omega) = (1/\omega)\mathrm{Im}\, k(\omega)$. Now, the FDT theorem gives the thermal noise spectrum:

$$S_x^{\rm th}(\omega) = \frac{4k_B\mathcal{T}}{k_R\omega}\frac{\phi(\omega)}{(1 - m\omega^2/k_R)^2 + \phi^2}. \tag{3.5}$$

Here, $k_R(\omega) \equiv \mathrm{Re}\, k(\omega)$, and

$$\phi(\omega) = \frac{\mathrm{Im}\, k(\omega)}{\mathrm{Re}\, k(\omega)} \tag{3.6}$$

is the so-called loss function. Note that $\phi = \tan\delta$, where δ is the angle by which strain lags behind stress. The loss function ϕ is a measure of the energy dissipation in the system. The rate at which energy is dissipated is $\overline{F_s \dot{x}}$. Then the energy dissipated per cycle by an anelastic spring is

$$\Delta\mathcal{E} = (2\pi/\omega)\overline{F_s \dot{x}}. \tag{3.7}$$

Taking $F_s = F_0 \cos\omega t$ and $x = x_0 \cos(\omega t - \delta)$, one finds

$$\Delta\mathcal{E} = \pi x_0 F_0 \sin\delta. \tag{3.8}$$

If δ is small than the total energy of spring vibration is $\mathcal{E} = \frac{1}{2} x_0 F_0$. Then for $\delta \ll 1$, one obtains

$$\phi = \frac{\Delta\mathcal{E}}{2\pi\mathcal{E}}. \tag{3.9}$$

For small ϕ (which is usually the case for the internal friction in materials used in detectors like LIGO), it is customary to neglect the frequency dependence of k_R. Then one can write $k(\omega) = k[1 + i\phi(\omega)]$, where $k = m\omega_0^2$ is a constant (and ω_0 is the resonant frequency). Though this is a good approximation for many practical reasons, in general k_R must be frequency-dependent because real and imaginary parts of $k(\omega)$ are related via the Kramers-Kronig relations.

3.2 Simple models

Here we consider some simple models of anelasticity in solids. Neither of them gives a full description of the behavior of a real material, but nevertheless they are useful from the didactic point of view.

3.2.1 Perfect elastic solid

The mechanical model of perfect elastic solid is a lossless spring. In this case $\sigma = E\epsilon$, so $\phi = 0$. There is no dissipation and no thermal noise.

3.2.2 Maxwell solid

The mechanical model of Maxwell solid is a lossless spring in series with a dashpot. The dashpot provides a source of viscous friction with $\sigma = \eta\dot{\epsilon}$. Then for Maxwell solid stress and strain are related by equation

$$\dot{\epsilon} = E^{-1}\dot{\sigma} + \eta^{-1}\sigma. \tag{3.10}$$

This equation shows that for a constant strain, stress decays exponentially. On the other hand, for a constant stress, strain increases linearly, which is a very wrong description for crystalline solids. Going to the frequency domain, one obtains

$$E(\omega) = \frac{\sigma(\omega)}{\epsilon(\omega)} = \frac{i\omega\eta E}{E + i\omega\eta} = \frac{\omega^2\eta^2 E + i\omega\eta E^2}{E^2 + \omega^2\eta^2} \tag{3.11}$$

and $\phi(\omega) = E/(\eta\omega)$.

3.2.3 Voigt-Kelvin solid

The mechanical model corresponding to Voigt-Kelvin anelastic solid consists of a lossless spring and a dashpot in parallel, which corresponds to a damped harmonic oscillator. The relation between stress and strain reads

$$\eta\dot{\epsilon} + E\epsilon = \sigma. \tag{3.12}$$

For a constant stress σ_0, strain changes exponentially with the decay time η/E from its initial value ϵ_0 to the equilibrium (Hooke) value σ_0/E. For a constant strain, stress is also constant, like in Hooke's law. This is a good description for materials like cork, but it is not suitable for metals. In the frequency domain, one has

$$E(\omega) = E + i\eta\omega, \qquad \phi(\omega) = (\eta/E)\omega. \tag{3.13}$$

Substituting this ϕ into Eq. (3.5), we find

$$S_x^{\rm th}(\omega) = \frac{4k_B\mathcal{T}\eta}{(m\omega^2 - E)^2 + \eta^2\omega^2}. \tag{3.14}$$

This is the same as Eq. (2.7) for a damped harmonic oscillator with $E \leftrightarrow k$ and $\eta \leftrightarrow \gamma$.

3.2.4 Standard anelastic solid

Though the model of standard anelastic solid (SAS) does not gives a complete account of properties of real metals, it describes quite well basic mechanisms responsible for anelastic effects. In fact, if a dissipation mechanism has characteristic relaxation times for strain upon a constant stress and for stress upon a constant strain, then the SAS model gives an adequate description. The corresponding mechanical model consists of a spring in parallel with a Maxwell unit (which is a spring in series with a dashpot). Let E_1 and E_2 be the Young moduli of the separate spring and of the spring in the Maxwell unit, respectively, and η be the dashpot constant, as usual. Then stress and strain are related by the following equation:

$$\frac{E_2}{\eta}\sigma + \dot{\sigma} = \frac{E_1E_2}{\eta}\epsilon + (E_1 + E_2)\dot{\epsilon}. \tag{3.15}$$

For a constant strain ϵ_0, stress decays exponentially from its initial value σ_0 to the equilibrium (Hooke) value $E_1\epsilon_0$:

$$\sigma(t) = E_1\epsilon_0 + (\sigma_0 - E_1\epsilon_0)e^{-t/\tau_\epsilon}, \qquad \tau_\epsilon = \eta/E_2. \tag{3.16}$$

Analogously, for a constant stress σ_0, strain decays exponentially from its initial value ϵ_0 to the equilibrium (Hooke) value σ_0/E_1:

$$\epsilon(t) = \frac{\sigma_0}{E_1} + \left(\epsilon_0 - \frac{\sigma_0}{E_1}\right)e^{-t/\tau_\sigma}, \qquad \tau_\sigma = \frac{E_1 + E_2}{E_1E_2}\eta. \tag{3.17}$$

Then Eq. (3.15) can be rewritten in the following form

$$\sigma + \tau_\epsilon\dot{\sigma} = E_R(\epsilon + \tau_\sigma\dot{\epsilon}), \tag{3.18}$$

where $E_R \equiv E_1$ is called the relaxed Young modulus. Transforming to the frequency domain, one obtains

$$(1 + i\omega\tau_\epsilon)\sigma(\omega) = E_R(1 + i\omega\tau_\sigma)\epsilon(\omega). \tag{3.19}$$

Then the complex Young modulus is given by

$$E(\omega) = E_R \frac{1 + i\omega\tau_\sigma}{1 + i\omega\tau_\epsilon} = E_R \frac{(1 + \omega^2\tau_\sigma\tau_\epsilon) + i\omega(\tau_\sigma - \tau_\epsilon)}{1 + \omega^2\tau_\epsilon^2}. \tag{3.20}$$

It is easy to see that

$$E(\omega) \approx \begin{cases} E_R, & \omega \ll 1 \\ E_U, & \omega \gg 1, \end{cases} \tag{3.21}$$

where $E_U \equiv E_1 + E_2$ is called the unrelaxed Young modulus. The loss function has the form

$$\phi(\omega) = \frac{\omega(\tau_\sigma - \tau_\epsilon)}{1 + \omega^2\tau_\sigma\tau_\epsilon} = \Delta \frac{\omega\bar{\tau}}{1 + \omega^2\bar{\tau}^2}, \tag{3.22}$$

where

$$\bar{\tau} = \sqrt{\tau_\sigma\tau_\epsilon} = \tau_\epsilon \sqrt{\frac{E_U}{E_R}}, \qquad \Delta = \frac{E_U - E_R}{\sqrt{E_U E_R}} = \frac{\tau_\sigma - \tau_\epsilon}{\sqrt{\tau_\sigma\tau_\epsilon}}. \tag{3.23}$$

One sees that $\phi \propto \omega$ for $\omega\bar{\tau} \ll 1$ and $\phi \propto \omega^{-1}$ for $\omega\bar{\tau} \gg 1$. The loss function has its maximum $\phi = \Delta/2$ at $\omega = \bar{\tau}^{-1}$. This is called the Debye peak. This behavior is characteristic for processes with exponential relaxation of stress and strain. $\bar{\tau}$ is the characteristic relaxation time and Δ is the relaxation strength.

Thermoelastic damping mechanism Zener pointed out that the SAS model with

$$\phi(\omega) = \Delta \frac{\omega\bar{\tau}}{1 + \omega^2\bar{\tau}^2}, \tag{3.24}$$

is suitable for describing processes in which the relaxation of stress and strain is related to a diffusion process. One example of such a process is the so-called thermoelastic damping. Consider a specimen which is a subject to a deformation in such a way that one part of it expands and the other contracts (e.g., a wire of a pendulum which bends near the top while the pendulum swings). The temperature increases in the contracted part and decreases in the expanded part. The resulting thermal diffusion leads to the dissipation of energy. This anelastic effect can be described by the SAS model with the thermal relaxation strength and relaxation time given by

$$\Delta = \frac{E_U \mathcal{T} \alpha^2}{C_v}, \qquad \bar{\tau} \simeq \frac{d^2}{D}, \tag{3.25}$$

where $\mathcal{T}$ is the temperature, α is the linear thermal expansion coefficient, C_v is the specific heat per unit volume, d is the characteristic distance heat must flow, and D is the thermal diffusion coefficient, $D = \varrho/C_v$, where ϱ is the thermal conductivity. For a cylindrical wire of diameter d, the frequency of the Debye peak is

$$\bar{f} = \frac{1}{2\pi\bar{\tau}} \simeq 2.6 \frac{D}{d^2}. \tag{3.26}$$

3.3 Boltzmann's superposition principle

While the SAS has certain general features in common with actual solids, it does not reproduce precisely the behavior of any real metal. Simple models considered above can be generalized by a theory which only assumes that the relation between stress and strain is linear. This assumption

was expressed by Boltzmann in the form of a superposition principle: If the deformation $x_1(t)$ was produced by the force $F_1(t)$ and the deformation $x_2(t)$ was produced by the force $F_2(t)$, then the force $F_1(t) + F_2(t)$ will produce the deformation $x_1(t) + x_2(t)$. On the other hand, the deformation can be regarded as the independent variable. In this case the superposition principle states: If the force $F_1(t)$ is required to produce the deformation $x_1(t)$ and the force $F_2(t)$ is required to produce the deformation $x_2(t)$, then the force $F_1(t) + F_2(t)$ will be required to produce the deformation $x_1(t) + x_2(t)$.

Let us introduce the quantity $\lambda(t)$ which is called the creep function and is the deformation resulting from the sudden application at $t = 0$ of a constant force of magnitude unity. During an infinitesimal interval from t to $t + dt$, the applied force $F(t)$ can be approximated by a constant force of magnitude $\dot{F}dt$. Then the superposition principle gives

$$x(t) = \int_{-\infty}^{t} \lambda(t - t')\dot{F}(t')dt'. \tag{3.27}$$

Conversely, we may regard the deformation as a specified function of time. Let us define the quantity $\kappa(t)$ which is called the stress function and is the force which must be applied in order to produce the step-function deformation $x(t) = \Theta(t)$ (here $\Theta(t)$ is 1 for $t \geq 0$ and 0 for $t < 0$). Then the linear relationship is

$$F(t) = \int_{-\infty}^{t} \kappa(t - t')\dot{x}(t')dt'. \tag{3.28}$$

The relation between the creep function and the strain function is rather complicated; in general they satisfy the following inequality

$$\lambda(t)\kappa(t) \leq 1. \tag{3.29}$$

For constant $\kappa(t) = k$ we recover Hooke's law $F(t) = kx(t)$ and then $\lambda(t) = k^{-1}$. Integrating by parts in Eq. (3.28), we obtain another expression of the superposition principle,

$$F(t) = \int_{-\infty}^{t} f(t - t')x(t')dt', \tag{3.30}$$

where

$$f(t) = \kappa(0)\delta(t) + \dot{\kappa}(t). \tag{3.31}$$

The relationship between the force and the deformation becomes very simple in the frequency domain. Toward this end we introduce the functions

$$f_p(t) = f(t)\Theta(t), \qquad \kappa_p(t) = \kappa(t)\Theta(t), \qquad \lambda_p(t) = \lambda(t)\Theta(t), \tag{3.32}$$

which are zero for $t < 0$. Using these functions, one can expand the upper integration limit in Eqs. (3.27), (3.28), and (3.30) to ∞. Then we just can use the fact that a convolution in the time domain is a product in the frequency domain. This gives

$$F(\omega) = i\omega\kappa_p(\omega)x(\omega) = f_p(\omega)x(\omega). \tag{3.33}$$

Thus the Fourier transform of the stress function is simply related to the complex spring constant of Eq. (3.2):

$$k(\omega) = f_p(\omega) = i\omega\kappa_p(\omega). \tag{3.34}$$

3.3.1 Example: Standard anelastic solid

For the SAS the stress function is given by

$$\kappa(t) = E_R + (E_U - E_R)e^{-t/\tau_\epsilon}. \tag{3.35}$$

It is straightforward to see that this function leads to the first-order differential equation of the form (3.18). Then we find the function $f(t)$,

$$f(t) = E_U\delta(t) - \frac{E_U - E_R}{\tau_\epsilon}e^{-t/\tau_\epsilon}, \tag{3.36}$$

and the complex string constant,

$$k(\omega) = \int_0^\infty f(t)e^{-i\omega t}dt = E_U - \frac{E_U - E_R}{1 + i\omega\tau_\epsilon}. \tag{3.37}$$

This can be rewritten in the form

$$k(\omega) = E_R\frac{1 + i\omega\tau_\sigma}{1 + i\omega\tau_\epsilon} = E_R\frac{(1 + \omega^2\tau_\sigma\tau_\epsilon) + i\omega(\tau_\sigma - \tau_\epsilon)}{1 + \omega^2\tau_\epsilon^2}. \tag{3.38}$$

which coincides with Eq. (3.20).

4 Calculation of the thermal noise spectrum for a pendulum suspension

For a point mass m attached to an anelastic spring with the complex spring constant $k(\omega)$, we found a simple result

$$Z(\omega) = \frac{k(\omega) - m\omega^2}{i\omega},$$

which can be used in the FDT to derive the thermal noise spectrum $S_x^{\rm th}(\omega)$ as given by Eq. (3.5). However, the question is how to find the thermal noise spectrum for more complicated systems, e.g., for pendulum suspensions of test masses in interferometric gravitational-wave detectors like LIGO.

In the literature we can find two different approaches: the "direct" application of the FDT to the whole system and the method which is based on decomposing a complicated system into a set of normal modes. Below, we describe briefly both of these approaches.

4.1 The direct approach

In brief, this method can be described as follows. First, one should write equations of motion for the whole system and find the impedance $Z(\omega)$. Then the FDT provides the thermal noise spectrum:

$$S_x^{\rm th}(\omega) = \frac{4k_BT}{\omega^2}{\rm Re}\,[1/Z(\omega)]. \tag{4.1}$$

The impedance $Z(\omega)$ contains the information about resonances of the system. The dissipation enters by taking the Young moduli of the materials to be complex:

$$E(\omega) = [1 + i\phi(\omega)]\,{\rm Re}\,E(\omega), \tag{4.2}$$

or, for simplicity, $E(\omega) = E_0[1 + i\phi(\omega)]$, where E_0 is a constant. The loss function $\phi(\omega)$ is obtained from experiments on the anelasticity of materials used in the system (e.g., on the suspension wires). Of course, the resulting noise spectrum $S_x^{\rm th}(\omega)$ depends very much on what form of $\phi(\omega)$ is used.

4.2 The normal-mode decomposition

The normal-mode decomposition is a more traditional approach. Consider, for example, an one-dimensional system of linear mass density $\rho(z)$ and total length L, which is described in terms of the normal modes $\psi_n(z)$. These modes satisfy the orthonormality relation,

$$\int_0^L \rho(z)\psi_m(z)\psi_n(z)dz = \delta_{mn}, \tag{4.3}$$

and an arbitrary displacement $x(z,t)$ can be decomposed as

$$x(z,t) = \sum_n \psi_n(z)q_n(t). \tag{4.4}$$

Here, $q_n(t)$ are the mode coordinates which satisfy

$$\ddot{q}_n + \omega_n^2 q_n = F_n(t), \tag{4.5}$$

where ω_n are the resonance frequencies of the modes, and

$$F_n(t) = \int_0^L f(z,t)\psi_n(z)dz \tag{4.6}$$

is the generalized force produced by the force density $f(z,t)$ applied to the system.

This decomposition effectively replaces the complicated system by a collection of oscillators, and each of them satisfies

$$[-\omega^2 + \omega_n^2(\omega)]q_n(\omega) = F_n(\omega). \tag{4.7}$$

The dissipation is included by taking

$$\omega_n^2(\omega) = \omega_n^2[1 + i\phi_n(\omega)], \tag{4.8}$$

where $\phi_n(\omega)$ are the loss functions. Then we can write

$$q_n(\omega) = \frac{F_n(\omega)}{-\omega^2 + \omega_n^2 + i\omega_n^2\phi_n(\omega)}. \tag{4.9}$$

Let us assume that the force is applied at the end of the system $z = L$, such that $f(z,t) = F(t)\delta(z-L)$. Then the generalized forces are $F_n(t) = F(t)\psi_n(L)$, and we can substitute Eq. (4.9) into the Fourier transform of Eq. (4.4) to obtain

$$x(L,\omega) \equiv x(\omega) = \sum_n \frac{\psi_n^2(L)}{-\omega^2 + \omega_n^2 + i\omega_n^2\phi_n(\omega)}F(\omega). \tag{4.10}$$

This gives the admittance of the system in the form

$$Y(\omega) = 1/Z(\omega) = \sum_n \frac{i\omega\psi_n^2(L)}{-\omega^2 + \omega_n^2 + i\omega_n^2\phi_n(\omega)}. \tag{4.11}$$

Then the FDT can be used to obtain the spectral density of thermal fluctuations at $z = L$:

$$S_x^{\rm th}(\omega) = \frac{4k_B\mathcal{T}}{\omega}\sum_n \frac{\psi_n^2(L)\omega_n^2\phi_n(\omega)}{(\omega_n^2 - \omega^2)^2 + \omega_n^4\phi_n^2}. \tag{4.12}$$

This can be written as a sum

$$S_x^{\rm th}(\omega) = \sum_n S_n^{\rm th}(\omega) \tag{4.13}$$

over the contributions

$$S_n^{\rm th}(\omega) = \frac{4k_B\mathcal{T}}{\omega}\frac{k_n^{-1}\phi_n(\omega)}{(1-m_n\omega^2/k_n)^2+\phi_n^2} = \frac{4k_B\mathcal{T}}{\omega}\frac{m_n^{-1}\omega_n^2\phi_n(\omega)}{(\omega_n^2-\omega^2)^2+\omega_n^4\phi_n^2} \tag{4.14}$$

of independent oscillators labeled by the index n. Each of these oscillators consists of a point mass $m_n = [\psi_n(L)]^{-2}$ attached to an anelastic spring with the complex spring constant $k_n(\omega) = k_n[1 + i\phi_n(\omega)]$, such that the resonant angular frequencies are $\omega_n = \sqrt{k_n/m_n}$. So, in order to obtain the thermal noise spectrum one needs to find all the normal modes, their effective masses, resonant frequencies, and loss functions.

4.3 Modes of a pendulum suspension

The most important modes of a pendulum suspension are the pendulum mode, the rocking mode, and the violin modes. We will not consider here the rocking mode because for multi-loop suspensions the rocking motion of the test mass is essentially suppressed. The loss function of each mode depends on the type of mode and on anelastic properties of the pendulum wire.

4.3.1 The pendulum mode

For the pendulum mode, we will assume that the mass of the wire is much smaller than the mass of the bob (which is the test mass) and that the bob is attached near its center of mass. Also, the angle by which the pendulum swings is considered to be very small. Then the pendulum may be modelled as an oscillator of the resonant angular frequency

$$\omega_{\rm p} = \sqrt{g/L}, \tag{4.15}$$

where g is the acceleration due to the Earth gravity field, and L is the pendulum length.

The energy of the pendulum consists of two parts: the gravitational energy $\mathcal{E}_{\rm gr}$ and the elastic energy $\mathcal{E}_{\rm el}$ due to the bending of the wire. The gravitational energy is lossless; provided that all the losses due to interactions with the external world (friction in the residual gas, dumping by eddy currents, recoil losses into the seismic isolation system, friction in the suspension clamps, etc.) are made insignificant by careful experimental design, the assumption is made that the losses are dominated by internal friction in the wire material. Consequently, $\Delta\mathcal{E} = \Delta\mathcal{E}_{\rm el}$. Usually, $\mathcal{E}_{\rm gr} \gg \mathcal{E}_{\rm el}$, so we obtain for the pendulum-mode loss function:

$$\phi_{\rm p} = \frac{\Delta\mathcal{E}}{2\pi\mathcal{E}_{\rm tot}} = \frac{\Delta\mathcal{E}_{\rm el}}{2\pi(\mathcal{E}_{\rm el}+\mathcal{E}_{\rm gr})} \approx \frac{\Delta\mathcal{E}_{\rm el}}{2\pi\mathcal{E}_{\rm el}}\frac{\mathcal{E}_{\rm el}}{\mathcal{E}_{\rm gr}}. \tag{4.16}$$

Note that

$$\phi_{\rm w} = \frac{\Delta\mathcal{E}_{\rm el}}{2\pi\mathcal{E}_{\rm el}} \tag{4.17}$$

is the loss function for the wire itself which occurs due to anelastic effects in the wire material. Then we obtain

$$\phi_{\rm p} = \xi_{\rm p}\phi_{\rm w}, \tag{4.18}$$

where $\xi_{\rm p}$ is the ratio between the elastic energy and the gravitational energy for the pendulum mode,

$$\xi_{\rm p} = \left(\frac{\mathcal{E}_{\rm el}}{\mathcal{E}_{\rm gr}}\right)_{\rm p}. \tag{4.19}$$

The lossless gravitational energy of the pendulum is

$$\mathcal{E}_{\rm gr} = \frac{1}{2}M\omega_{\rm p}^2 L^2\theta_m^2 = \frac{1}{2}MgL\theta_m^2, \tag{4.20}$$

where M is the pendulum mass and θ_m is the maximum angle of swing. The elastic energy depends on how many wires are used and how they are attached to the pendulum. For one wire, the fiber in the pendulum mode will bend mostly near the top, with the bending elastic energy

$$\mathcal{E}_{\rm el} = \frac{1}{4}\sqrt{TEI}\theta_m^2. \tag{4.21}$$

Here, T is the tension force in the wire ($T = Mg$ for one wire), E is the Young modulus of the wire material, and I is the moment of inertia of the wire cross section ($I = \frac{1}{2}\pi r^4$ for a cylindrical wire of radius r). Using these results, one finds for a single-wire pendulum:

$$\xi_{\rm p} = \frac{\sqrt{TEI}}{2MgL} = \frac{1}{2L}\sqrt{\frac{EI}{Mg}} = \frac{1}{2L}\sqrt{\frac{EI}{T}}. \tag{4.22}$$

This result can be easily generalized for the case when the test muss is suspended by N wires. Then the elastic energy $\mathcal{E}_{\rm el}$ of Eq. (4.21) should be multiplied by N and the tension in each wire becomes $T = Mg/N$. Then

$$\xi_{\rm p} = \frac{N\sqrt{TEI}}{2MgL} = \frac{1}{2L}\sqrt{\frac{EIN}{Mg}} = \frac{1}{2L}\sqrt{\frac{EI}{T}}. \tag{4.23}$$

In Eq. (4.23) we assumed that all the wires are in one plane: a plane through the center of mass of the pendulum, whose normal is parallel to the direction of swing. (Note that in such an configuration one should take into account the rocking mode of the test mass.) In this arrangement, the pendulum mode causes bending of the wires mostly at the top. If one uses a number of wire loops along the test mass length, then the rocking mode is essentially suppressed and the wires bend both at the top and the bottom. Therefore, the bending elastic energy of the multi-loop configuration is given by multiplying the result of Eq. (4.21) by $2N$,

$$\mathcal{E}_{\rm el} = \frac{N}{2}\sqrt{TEI}\theta_m^2. \tag{4.24}$$

Then the energy ratio is

$$\xi_{\rm p} = \frac{N\sqrt{TEI}}{MgL} = \frac{1}{L}\sqrt{\frac{EIN}{Mg}} = \frac{1}{L}\sqrt{\frac{EI}{T}}. \tag{4.25}$$

The contribution of the pendulum mode to the thermal noise spectrum is obtained from Eq. (4.14) by taking $m_n = M$, $k_n = Mg/L$, $\omega_n = \omega_{\rm p}$ and $\phi_n = \phi_{\rm p} = \xi_{\rm p}\phi_{\rm w}$. This gives

$$S_{\rm p}^{\rm th}(\omega) = \frac{4k_B\mathcal{T}}{\omega M}\frac{\omega_{\rm p}^2\phi_{\rm p}(\omega)}{(\omega_{\rm p}^2-\omega^2)^2+\omega_{\rm p}^4\phi_{\rm p}^2}. \tag{4.26}$$

For LIGO suspensions, $f_{\rm p} = \omega_{\rm p}/2\pi$ is about 1 Hz. This is much below the working frequency range (near 100 Hz), so we may assume $\omega_{\rm p}/\omega \ll 1$. Also, the loss function is very small, $\phi_{\rm p} < 10^{-5}$. Then the pendulum-mode contribution to the thermal noise spectrum is

$$S_{\rm p}^{\rm th}(\omega) \simeq \frac{4k_B\mathcal{T}\omega_{\rm p}^2\phi_{\rm p}(\omega)}{M\omega^5} = \frac{4k_B\mathcal{T}}{L^2}\sqrt{\frac{gEIN}{M^3}}\frac{\phi_{\rm w}(\omega)}{\omega^5}. \tag{4.27}$$

4.3.2 The violin modes

The angular frequency of the nth violin mode ($n = 1, 2, 3, \ldots$) is given by

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}}\left[1 + \frac{2}{k_eL} + \frac{1}{2}\left(\frac{n\pi}{k_eL}\right)^2\right], \tag{4.28}$$

where L is the length of the wire, T is the tension force, ρ is the linear mass density of the wire, and

$$k_e \simeq \sqrt{\frac{T}{EI}}. \tag{4.29}$$

In the violin mode the wire bends near both ends in a similar way. The bending occurs over the characteristic distance scale $k_e^{-1} \simeq \sqrt{EI/T}$, the same as in the pendulum mode. For $k_e^{-1} \ll L$, which is a very good estimation for heavily loaded thin wires like in LIGO, one have approximately,

$$\omega_n \simeq \frac{n\pi}{L}\sqrt{\frac{T}{\rho}}. \tag{4.30}$$

This is just the angular frequency of the nth vibrational mode of an ideal spring.

It can be shown that for the nth violin mode, the loss function is

$$\phi_n = \xi_n\phi_{\rm w}, \qquad \xi_n = \left(\frac{\mathcal{E}_{\rm el}}{\mathcal{E}_{\rm gr}}\right)_n, \tag{4.31}$$

where the energy ratio is

$$\xi_n = \frac{2}{k_eL}\left(1 + \frac{n^2\pi^2}{2k_eL}\right) \simeq \frac{2}{L}\sqrt{\frac{EI}{T}}\left(1 + \frac{1}{2L}\sqrt{\frac{EI}{T}}n^2\pi^2\right). \tag{4.32}$$

Since $k_eL \gg 1$, for first several modes the energy ratio is approximately

$$\xi_n \simeq \xi_{\rm v} = \frac{2}{L}\sqrt{\frac{EI}{T}}. \tag{4.33}$$

This expression takes into account only the contribution to the elastic energy due to wire bending near the top and the bottom. For higher violin modes, one should also consider the contribution due to wire bending along its length, which leads to Eq. (4.32).

For the one-loop suspension configuration, the elastic energy of the lowest violin modes is about twice of that for the pendulum mode (for the last one the wires bend only at the top while for the former ones the wires bend at both ends). In the multi-loop configuration, the elastic energy of the

lowest violin modes and of the pendulum mode is approximately the same. On the other hand, the gravitational energy of the pendulum mode is by a factor of 2 larger than that of a violin mode. For the violin modes of each wire, the gravitational energy is $\frac{1}{4}TL\theta_m^2$. Then for N wires,

$$(\mathcal{E}_{\rm gr})_{\rm v} = \frac{1}{4}NTL\theta_m^2 = \frac{1}{4}MgL\theta_m^2. \tag{4.34}$$

This is just one half of the gravitational energy for the pendulum mode, $(\mathcal{E}_{\rm gr})_{\rm p} = \frac{1}{2}MgL\theta_m^2$ (cf. Eq. (4.20)). This explains the difference between the loss functions for the pendulum mode and for the violin modes: $\xi_{\rm v} \simeq 4\xi_{\rm p}$ for the one-loop configuration and $\xi_{\rm v} \simeq 2\xi_{\rm p}$ for the multi-loop configuration.

The effective mass of the nth violin mode is

$$m_n = [\psi_n(L)]^{-2} = \frac{1}{2}NM\left(\frac{\omega_n}{\omega_{\rm p}}\right)^2 \simeq \frac{\pi^2 M^2}{2\rho L}n^2, \tag{4.35}$$

where we took expression (4.30) for ω_n and $T = Mg/N$. This effective mass arises because the violin vibrations of the wire cause only a tiny recoil of the test mass M. The contribution of the violin modes to the thermal noise spectrum is given by

$$S_{\rm v}^{\rm th}(\omega) = \frac{4k_B\mathcal{T}}{\omega}\sum_{n=1}^{\infty}\frac{m_n^{-1}\omega_n^2\phi_n(\omega)}{(\omega_n^2-\omega^2)^2+\omega_n^4\phi_n^2}. \tag{4.36}$$

Typical values of $f_1 = \omega_1/2\pi$ are from 350 to 500 Hz. If we are interested in the thermal spectrum density near 100 Hz, we can assume $\omega^2 \ll \omega_n^2$. Then we have approximately

$$S_{\rm v}^{\rm th}(\omega) \simeq \frac{8k_B\mathcal{T}\omega_{\rm p}^2}{NM\omega}\sum_{n=1}^{\infty}\frac{\phi_n(\omega)}{\omega_n^4} \simeq \frac{8k_B\mathcal{T}N\rho^2L^3}{\pi^4 gM^3\omega}\sum_{n=1}^{\infty}\frac{\phi_n(\omega)}{n^4}. \tag{4.37}$$

One can see that the contributions of higher violin modes are very small due to the factor n^{-4} in the sum. Taking $\phi_n = \xi_n\phi_{\rm w}$ and using Eq. (4.32), we obtain

$$\sum_{n=1}^{\infty}\frac{\phi_n(\omega)}{n^4} = \frac{2}{k_eL}\left(\frac{\pi^4}{90}+\frac{\pi^4}{12k_eL}\right)\phi_{\rm w}(\omega) \simeq \frac{\pi^4}{45L}\sqrt{\frac{EI}{T}}\phi_{\rm w}(\omega). \tag{4.38}$$

Here, we assumed $k_eL \gg 1$. Finally, we substitute (4.38) into (4.37) and find the following expression for the violin-mode contribution to the thermal noise spectrum,

$$S_{\rm v}^{\rm th}(\omega) \simeq \frac{8}{45}k_B\mathcal{T}\rho^2L^2\sqrt{\frac{EIN^3}{g^3M^7}}\frac{\phi_{\rm w}(\omega)}{\omega}. \tag{4.39}$$

5 Experiments on anelasticity effects for pendulum suspensions

5.1 Basic types of experiments

In order to predict the thermal noise fluctuations in pendulum suspensions, two basic types of experiments are performed:

1. Investigations of anelastic properties of wires made of various materials, in order to determine the wire loss function $\phi_{\rm w}(\omega)$.

2. Measurements of quality factors ($Q = \phi^{-1}$ at a resonance) for the pendulum and violin modes of actual suspensions, in order to verify the relationships

$$\phi_{\rm p}(\omega) = \xi_{\rm p}\phi_{\rm w}(\omega), \qquad \phi_{\rm v}(\omega) = \xi_{\rm v}\phi_{\rm w}(\omega). \tag{5.1}$$

Early experiments showed serious discrepancy between the measured quality factors and those predicted using Eq. (5.1). It was discussed that this discrepancy may happen due to stress-dependent effects. However, it was shown later that the internal losses of the wires are almost independent of the applied stress. Many recent experiments proved that the above discrepancy appears due to serious losses in the clamps. A smart design of clamps can be used to reduce these excess losses and then predictions of Eq. (5.1) are quite accurate. A very promising possibility is the use of monolithic or semi-monolithic suspensions. The design of clamps plays a crucial role in the reduction of the thermal noise of the test mass suspensions.

5.2 Internal losses in wire materials

A number of experiments were performed to study internal losses of various wire materials (e.g., steel, tungsten, fused quartz, and some others). The main drawback of many of these experiments is a small number of frequencies for which $\phi_{\rm w}$ was measured. Also, there are serious discrepancies between results of different experiments. Therefore, the exact frequency dependence of $\phi_{\rm w}$ is still unclear for many materials. Below, we briefly review results of some recent experiments.

Kovalik and Saulson, 1993 Method: Quality factors were measured for resonances of freely suspended wires. Materials: Tungsten, silicon, sapphire, fused quartz. Results: Insignificant frequency dependence for tungsten; for fused quartz, measured $\phi_{\rm w}$ are above those predicted by the thermoelastic damping (TED) for some frequencies and near TED for others; sapphire and silicon showed behavior consistent with TED.

Saulson et al., 1994 Method: Quality factors were measured for an inverted pendulum of tunable length. Material: Free-Flex cross-spring flexure made of crossed steel strips. Results: In agreement with a frequency-independent $\phi_{\rm w}$.

Gillespie and Raab, 1994 Method: Quality factors were measured for resonances of freely suspended wires. Material: Steel music wires. Results: A constant value of $\phi_{\rm w}$ for low frequencies (from 30 to 150 Hz). For higher frequencies (from 150 Hz to 2 kHz) $\phi_{\rm w}$ increases with ω, like TED predicts, but the measured value $\phi_{\rm meas}$ is well above $\phi_{\rm TED}$. These results may be explained by the formula $\phi_{\rm meas} = \phi_{\rm TED} + \phi_{\rm ex}$, where $\phi_{\rm ex}$ is a frequency-independent excess loss.

Rowan et al., 1997 Method: Quality factors were measured for resonances of ribbons fixed at one end. Material: Fused quartz ribbons. Results: Data were obtained for 5 resonances in the range from 6 to 160 Hz. $\phi_{\rm meas}$ is well above $\phi_{\rm TED}$ for lower frequencies (below 30 Hz), and in agreement with TED for higher frequencies (above 80 Hz).

Dawid and Kawamura, 1997 Method: Quality factors were measured for the violin modes of wires fixed at both ends in a "guitar"-type apparatus. Materials: Invar, titanium, steel, tungsten and several other metals. Results: $\phi_{\rm meas}^{-1}$ was proportional to $\sqrt{T}$, in accordance with the formula $\phi_{\rm v} = (2/L)\sqrt{EI/T}\phi_{\rm w}$ for frequency-independent $\phi_{\rm w}$.

Huang and Saulson, 1998 Method: Quality factors were measured for resonances of freely suspended wires. Materials: Steel and tungsten. For steel, $\phi_{\rm meas}$ coincides with the predictions of TED (the characteristic Debye-peak frequency dependence). Some differences were found between properties of annealed wires ($\phi_{\rm meas}$ slightly above $\phi_{\rm TED}$) and "curly" wires ($\phi_{\rm meas}$ slightly below $\phi_{\rm TED}$). The difference can be explained by modifications of thermal properties. For tungsten wires, $\phi_{\rm meas}$ only slightly increases with frequency; the loss function increases with the wire diameter, as should be for TED at frequencies well below $\bar{f}$.

6 Conclusions

It is seen that predictions of the spectral density for thermal fluctuations in pendulum suspensions depend strongly on the type of the dissipation mechanism. Sources of external losses (friction in the residual gas, dumping by eddy currents, recoil losses into the seismic isolation system, friction in the suspension clamps, etc.) should be eliminated by careful experimental design. In particular, results of many recent experiments show that excess losses in clamps may seriously deteriorate the quality factors of suspension resonances. When external losses are made sufficiently small, the main source of dissipation is the internal friction in the wires due to anelastic effects. The thermal noise spectrum depends on the form of the loss function. Unfortunately, the exact frequency dependence of the wire loss function $\phi_{\rm w}(\omega)$ is not yet completely understood. In many experiments $\phi_{\rm w}$ was measured only at few frequencies and experimental uncertainty of results was often quite large. Moreover, there is a contradiction between results of different experiments. Therefore, it is very difficult to make certain conclusions about the behavior of $\phi_{\rm w}(\omega)$. In particular, it is unclear if clamp losses are negligible in experiments with freely suspended wires, as is usually assumed. Certainly, there is a room for more experiments on anelastic properties of wires, in order to clarify the issue of internal friction in the frequency range of interest for gravitational-wave detection.

Acknowledgment

This work would be impossible without great help and encouragement by Malik Rakhmanov. I thank him for long hours of illuminating discussions and for encouraging me to enter the realm of thermal noise and anelasticity.

Appendix: Correlation function and spectral density

Consider a system characterized by some quantity α (e.g., position or velocity). For stationary processes, the correlation function is

$$\rho_\alpha(t) = \langle \alpha(\tau)\alpha(\tau + t)\rangle,$$

where the average is over a statistical ensemble. Using the ergodic theorem, this can be replaced by the time average,

$$\rho_\alpha(t) = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} \alpha(t')\alpha(t' + t)dt'.$$

Now, define the function

$$\alpha_T(t) = \begin{cases} \alpha(t), & t \in [-T, T] \\ 0, & \text{other} \end{cases}$$

and its Fourier transform

$$\alpha_T(\omega) = \int_{-\infty}^{\infty} \alpha_T(t) e^{i\omega t} dt.$$

The definition of the spectral density is

$$S_\alpha(\omega) = \lim_{T\to\infty} \frac{|\alpha_T(\omega)|^2}{\pi T}.$$

It is easy to see that the correlation function $\rho_\alpha(t)$ and the spectral density $S_\alpha(\omega)$ are related via the Fourier transform:

$$\rho_\alpha(t) = \frac{1}{2}\int_{-\infty}^{\infty} S_\alpha(\omega) e^{-i\omega t} d\omega, \qquad S_\alpha(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \rho_\alpha(t) e^{i\omega t} dt.$$

This result is known as the Wiener-Khinchin theorem.

Bibliography

The fluctuation-dissipation theorem was introduced in
H. B. Callen and T. A. Welton, "Irreversibility and Generalized Noise," Phys. Rev. **83**, 34 (1951);
H. B. Callen and R. F. Greene, "On a Theorem of Irreversible Thermodynamics," Phys. Rev. **86**, 702 (1952).
The theorem is discussed in a number of textbooks on statistical physics, for example,
L. E. Reichl, *A Modern Course in Statistical Physics* (Univ. Texas Press, Austin, 1980);
L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, Oxford, 1980).

The theory of anelasticity is discussed in
C. Zener, *Elasticity and Anelasticity of Metals* (Univ. Chicago Press, Chicago, 1948);
A. S. Novick and B. S. Berry, *Anelastic Relaxation in Crystalline Solids* (Academic Press, New York, 1972).
The theory of thermoelastic damping was presented by Zener:
C. Zener, "Theory of Internal Friction in Reeds," Phys. Rev. **52**, 230 (1937);
C. Zener, "General Theory of Thermoelastic Internal Friction," Phys. Rev. **53**, 90 (1938).

Thermal fluctuations of pendulum suspensions and related problems were discussed in many works. Some of them are listed below:
P. R. Saulson, "Thermal noise in mechanical experiments," Phys. Rev. D **42**, 2437 (1990);
G. I. González and P. R. Saulson, "Brownian motion of a mass suspended by an anelastic wire" J. Acoust. Soc. Am. **96**, 207 (1994);
J. E. Logan, J. Hough, and N. A. Robertson, "Aspects of the thermal motion of a mass suspended as a pendulum by wires," Phys. Lett. A **183**, 145 (1993);
A. Gillespie and F. Raab, "Thermal noise in the test mass suspensions of a laser interferometer gravitational-wave detector prototype," Phys. Lett. A **178**, 357 (1993);
J. Gao, L. Ju, and D. G. Blair, "Design of suspension systems for measurement of high-Q pendulums," Meas. Sci. Technol. **6**, 269 (1995);
V. B. Braginsky, V. P. Mitrofanov, and K. V. Tokmakov, "On the thermal noise from the violin modes of the test mass suspension in gravitational-wave antennae," Phys. Lett. A **186**, 18 (1994);
V. B. Braginsky, V. P. Mitrofanov, and K. V. Tokmakov, "Energy dissipation in the pendulum mode of the test mass suspension of a gravitational wave antenna," Phys. Lett. A **218**, 164 (1996);

G. Cagnoli, L. Gammaitoni, J. Kovalik, F. Marchesoni, and M. Punturo, "Suspension losses in low-frequency mechanical pendulums," Phys. Lett. A **213**, 245 (1996).

Experiments on internal friction in various types of wires (see Sec. 5.2) were reported in the following papers:
J. Kovalik and P. R. Saulson, "Mechanical loss in fibers for low-noise pendulums," Rev. Sci. Instrum. **64**, 2942 (1993);
P. R. Saulson, R. T. Stennins, F. D. Dumont, and S. E. Mock, "The inverted pendulum as a probe of anelasticity," Rev. Sci. Instrum. **65**, 182 (1994);
A. Gillespie and F. Raab, "Suspension losses in the pendula of laser interferometer gravitational-wave detectors," Phys. Lett. A **190**, 213 (1994);
S. Rowan, R. Hutchins, A. McLaren, N. A. Robertson, S. M. Twyford, and J. Hough, "The quality factor of natural fused quartz ribbons over a frequency range from 6 to 160 Hz," Phys. Lett. A **227**, 153 (1997);
D. J. Dawid and S. Kawamura, "Investigation of violin mode Q for wires of various materials," Rev. Sci. Instrum. **68**, 4600 (1997);
Y. L. Huang and P. R. Saulson, "Dissipation mechanisms in pendulums and their implications for gravitational wave interferometers," Rev. Sci. Instrum. **69**, 544 (1998).